Mass Formulas and Mass Inequalities for Reducible Unitary Multiplets*

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For irreducible unitary multiplets of elementary particles, the Gell-Mann–Okubo mass formula is equivalent to the assertion that the effective masses transform under SU(3) like a superposition of a singlet and the neutral member of an octet. We consider the implications of this assertion for reducible unitary multiplets. We find weaker mass formulas, which allow for the mixing between different irreducible components; moreover, we find that certain inequalities must be satisfied. Specific examples are described.

I. INTRODUCTION

THE "eightfold way"[1,2] groups strongly interacting particles into supermultiplets, normally corresponding to irreducible representations of SU(3). The masses within these supermultiplets would be degenerate were unitary symmetry exact; in reality, unitary symmetry is badly broken, and departures from degeneracy of several hundred MeV are common. These departures from the eightfold way are very accurately described by the Gell-Mann–Okubo mass formula[1,3]: For an irreducible supermultiplet of baryons, the masses are given by an expression of the form

$$M=a+bY+c[T(T+1)-Y^2/4]. \qquad (1)$$

For an irreducible supermultiplet of mesons, the squares of the masses are given by an expression of the same form, except that the term linear in Y is missing if the supermultiplet is transformed into itself by charge conjugation.

The mass formula is equivalent to the statement that the effective-mass Lagrangian is the sum of a term invariant under SU(3) and a term which transforms like the neutral ($T=Y=0$) member of a unitary octet. Although many explanations of the mass formula have been advanced, we treat it here as a given fact. We believe that the mass formula is better established than any of its derivations.

In this note, we discuss the form of the mass relations for reducible representations of SU(3), assuming that the effective mass Lagrangian transforms like the sum of an invariant and the neutral component of a unitary octet. This problem is of physical interest because there may occur cases where several irreducible representations are occupied with particles of the same spin and parity. If the masses of these supermultiplets are not too widely separated, particles with the same isospin-hypercharge assignments will mix with each other, and the independent mass formulas for the individual representations will no longer be valid.

For simplicity, we consider only reducible representations that are the sum of two irreducible components. We discuss the mass formulas that survive mixing; in addition we point out certain mass inequalities that have no counterpart in the case of irreducible representations. These arise for the following reason: The matrix that represents the effective-mass Lagrangian must be real and symmetric, as a consequence of Hermiticity and time-reversal invariance. We can find this matrix by solving a set of quadratic equations whose coefficients are the observed masses. The solution to these equations will automatically be symmetric, but may not necessarily be real. The condition that the roots be real imposes restrictions on the range of the masses; these are our inequalities.

We phrase our general discussion in terms of baryons; the reader should have little difficulty adapting it to mesons.

We distinguish three cases: (1) The two irreducible representations are equivalent. (2) The two irreducible representations are inequivalent and the product of one with the conjugate of the other contains the adjoint representation once. (3) The two irreducible representations are inequivalent and the product of one with the conjugate of the other does not contain the adjoint representation at all. Let us characterize irreducible representations of SU(3) as the transformations induced, in a three-dimensional complex vector space, on traceless symmetric tensors with p upper indices and q lower indices, and label them by the pair of integers (p,q). Then if the two irreducible representations are (p,q) and (p',q'), the first case occurs if $p=p'$ and $q=q'$. The second case occurs if $p=p'\pm1$ and $q=q'\pm1$, if $p=p'\pm1$ and $q=q'\mp2$, or if $p=p'\pm2$ and $q=q'\mp1$. The third case occurs otherwise.

(1) The two irreducible representations are equivalent. In this case every isospin-hypercharge multiplet that occurs in one representation has a matching multiplet in the other. For each such pair, the effective-mass Lagrangian is a quadratic form, characterized by a matrix of the form

$$\begin{pmatrix} M_1 & M_3 \\ M_3 & M_2 \end{pmatrix}, \qquad (2)$$

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[1] M. Gell-Mann, California Institute of Technology Synchotron Laboratory, Report No. 20, 1961 (unpublished).
[2] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).
[3] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

TABLE I. Mixing coefficients for simple representations.

Reducible representation	Mixing coefficient for overlapping isotopic multiplets				
	Σ	Ξ	N	Λ	Δ
$8\oplus 10$	1	1	$\cdots$	$\cdots$	$\cdots$
$8\oplus 27$	$(\frac{2}{3})^{1/2}$	1	1	$(\frac{3}{2})^{1/2}$	$\cdots$
$10\oplus 27$	$8^{1/2}$	$\sqrt{3}$	$\cdots$	$\cdots$	$(15)^{1/2}$

where the M_i are expressions of the form given in Eq. (1). The eigenvalues of this matrix are

$$\tfrac{1}{2}(M_1+M_2)\pm\tfrac{1}{2}[(M_1-M_2)^2+4M_3^2]^{1/2}. \tag{3a}$$

Thus, the mean masses of the pairs obey the Gell-Mann–Okubo formula. (This is independent of the octet form of the mixing.) The mass differences, on the other hand, are the lengths of vectors in the Euclidean plane which obey the formula, as vectors. For most of the interesting cases (e.g., the mixing of two octets), this last condition yields no equations for the mass differences. However, it always yields inequalities.

The mixing angle for the two multiplets is given by

$$\tan 2\theta = 2M_3/(M_1-M_2). \tag{3b}$$

Thus, the angle the mass-difference vector forms with the horizontal is twice the mixing angle.

Note that the entire geometric configuration which describes the mass differences and mixing angles is invariant under orthogonal transformations. This reflects our ignorance of which pair of supermultiplets would be the mass eigenstates in the limit of exact unitary symmetry. We choose a pair arbitrarily, but any orthonormal linear combination would do as well. (Thus we can always choose our basis so that one of the mixing angles is zero.)

(2) The two irreducible representations are inequivalent and the product of one with the conjugate of the other contains the adjoint representation once. In this case, the effective mass Lagrangian contains three parameters for each representation, but only one mixing parameter. The coefficients of the mixing term for the multiplets common to both representations are proportional to the coupling constants for an η to the two supermultiplets, and may be found in tables of SU(3) coefficients.[4] Their value for some simple representations are given in Table I. It may be shown that it is always possible to determine the six parameters that enter into the masses in the absence of mixing from the masses of those multiplets which only occur in one representation and from the mean masses of those multiplets common to both representations. Then one mass difference between a pair of common multiplets is sufficient to determine the magnitude of the mixing parameter. Thus we have one less relation among the masses than there would be if there were no mixing. Since the mixing parameter is the only parameter that has to be determined by solving a quadratic equation, we have only one independent inequality. It is the statement that the mass splitting between each mixed pair is larger than it would be in the absence of mixing. Since we know all the parameters in the effective-mass Lagrangian, we know all the mixing angles (to within a sign).

(3) The product of one representation with the conjugate of the other does not contain the adjoint representation at all. In this case, there is no mixing, and the Gell-Mann–Okubo formula applies to each multiplet independently.

II. EXAMPLES WITH MESONS

We use the symbols π, K, and η to denote states with hypercharge and isospin of the corresponding pseudoscalar mesons, and also to denote the squares of their masses. We always indicate by a prime the heavier member of a pair of mesons with the same quantum numbers. The vector mesons do not satisfy the Gell-Mann–Okubo formula for an irreducible representation of SU(3); the first three examples alternatively interpret the vector mesons as belonging to the reducible representations $1\oplus 8$,[5] $8\oplus 8$,[6] and $8\oplus 27$. The fourth example is concerned with the recently suggested "icosuplet" of bosons.[7]

$1\oplus 8$

There is no mass formula relating the masses of the four isotopic multiplets comprising a mixed singlet and octet, but the inequality,

$$\eta'\geq \tfrac{4}{3}K-\tfrac{1}{3}\pi\geq \eta, \tag{4}$$

must be satisfied. [One may attribute the discrepancy of the G.M.O. formula for pseudoscalar mesons to the presence of another unitary singlet η' mixing with $\eta(550)$. The above inequality puts η' above 570 MeV.)

The $\phi-\omega$ mixing model[5] of the nine vector mesons—$\bar{K}^*(885)$, $K^*(885)$, $\rho(750)$, $\omega(785)$, and $\phi(1020)$—treats them as the reducible representation $1\oplus 8$. The inequality is satisfied,

$$1.04\geq 0.85\geq 0.62 \quad (\text{in BeV}^2),$$

and the degree of mixing of ω and ϕ may be determined from the observed masses.[5]

$8\oplus 8$

Only one mass formula describes the masses of two mixed octets of mesons:

$$K+K'=\tfrac{3}{4}(\eta+\eta')+\tfrac{1}{4}(\pi+\pi'). \tag{5}$$

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[4] J. deSwart, Rev. Mod. Phys. 35, 916 (1963).

[5] J. J. Sakurai, Phys. Rev. Letters 9, 472 (1962).

[6] A. Balachandran, P. Freund, and C. Schumacher, Phys. Rev. Letters 12, 209 (1964).

[7] B. W. Lee, S. Okubo, and J. Schecter (to be published).

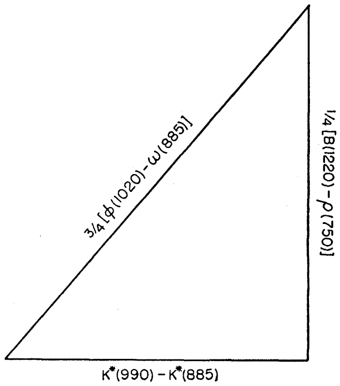

FIG. 1. The triangle describing the mixing of two possible octets of vector mesons. The sides of the triangle obey the Gell-Mann–Okubo formula, as vectors. The lengths of the sides are the differences in the squares of the masses of the associated pair of multiplets; the angles the sides make with the horizontal are twice the mixing angles. The angles obtained in this way agree with those of Freund (Ref. 9), within an arbitrary additive angle.

Since the mass differences must be lengths of vectors which themselves obey the Gell-Mann–Okubo formula, we also have the following triangle inequalities:

$$\begin{aligned} &K'-K+\tfrac{3}{4}(\eta'-\eta)\geq\tfrac{1}{4}(\pi'-\pi)\,,\\ &\tfrac{3}{4}(\eta'-\eta)+\tfrac{1}{4}(\pi'-\pi)\geq(K'-K)\,,\\ &\tfrac{1}{4}(\pi'-\pi)+(K'-K)\geq\tfrac{3}{4}(\eta'-\eta)\,. \end{aligned}\tag{6}$$

The suggestion has been made[5] that the vector mesons comprise two mixed octets composed of the nine usual states, the[8] $B(1220)$ (assumed to be a second $J=1^-$ isovector meson) and a second K^*. The above mass formula predicts the second K^* at 990 MeV. The two relative mixing angles may be determined from the physical masses, and turn out to be such as greatly to suppress the allowed decay scheme $K^*(990)\rightarrow K+\pi$.[9] The triangle inequalities are satisfied for this assignment.

Figure 1 shows the triangle associated with these masses. A triangle is determined (up to a Euclidean transformation) by the lengths of its sides; thus, if we set one of the mixing angles equal to zero, the other two are determined (up to a sign).

Another assignment, also suggested,[6] uses $K^*(730)$ for the lighter $T=\frac{1}{2}$, $Y=1$ vector meson. Then, the mass formula puts the heavier K^* at 1100 MeV, but the triangle inequalities are no longer satisfied.

$8\oplus 27$

As another possibility, we propose that the vector mesons may comprise the reducible representation $8\oplus 27$. Besides the nine usual vector mesons, and a second $T=1$, $Y=0$ vector meson again taken to be $B(1220)$, this representation contains the following unobserved $J=1^-$ isotopic multiplets:

(1) a second K^* with $T=\frac{1}{2}$, $Y=\pm 1$,
(2) a state with $T=\frac{3}{2}$, $Y=\pm 1$,
(3) a state with $T=1$, $Y=\pm 2$,
(4) a state with $T=2$, $Y=0$.

The masses of all the vector mesons are determined in terms of four coefficients for the unmixed Gell-Mann–Okubo formulas and one mixing parameter. Thus, everything is determined by the five known masses: $\rho(750)$, $B(1220)$, $\omega(785)$, $\phi(1020)$, and $K^*(885)$. We find that the mixing parameter so determined is real, and hence that the appropriate inequality is satisfied. The predicted masses of the unobserved multiplets are:

$$\begin{aligned} M(K^{*\prime})&=990\text{ MeV}\,,\\ M(T=\tfrac{3}{2},\ Y=\pm 1)&=1450\text{ MeV}\,,\\ M(T=1,\ Y=\pm 2)&=1040\text{ MeV}\,,\\ M(T=2,\ Y=0)&=1760\text{ MeV}\,. \end{aligned}$$

The existence of these unobserved states does not seem to be ruled out by present experiments. Perhaps the most interesting prediction is the existence of a doubly strange vector meson just above threshold.

The unmixed members of the 27-plet are forbidden to decay into two pseudoscalar mesons in the limit of exact unitary symmetry. This is because the 27-plet is a symmetric combination of two octets, and a $J=1^-$ state of two spinless mesons must be antisymmetric.

$10\oplus\overline{10}$

The existence of a 20-plet of meson resonances including a degenerate $T=1$ $\rho\pi$ resonance and an $\omega\pi$ resonance has been suggested.[7] Although involving two inequivalent representations of SU(3), this corresponds to an irreducible 20-dimensional representation of the group generated by SU(3) and charge conjugation. The Gell-Mann–Okubo formula is of the form

$$\mu^2=a+BY\quad(\text{for the }10)\,,\tag{7a}$$

and

$$\mu^2=a-BY\quad(\text{for the }\overline{10})\,.\tag{7b}$$

There is no mixing because $10\otimes 10$ does not contain 8; however, there is an isotopic multiplet ($T=1$, $Y=0$) common to both 10 and $\overline{10}$. So long as the effective masses are assumed to transform like $1\oplus 8$, these two states must remain degenerate. Thus, the 20-plet contains two $T=1$, $Y=0$ states that are equal in mass but opposite in G parity.

III. EXAMPLES WITH BARYON STATES

We use the symbols N, Ξ, Λ, and Σ to denote states with hypercharge and isospin of the corresponding baryons, and also to denote their masses. We also use the symbols Ω for $T=0$, $Y=2$, Δ for $T=\frac{3}{2}$, $Y=1$ and

[8] See, for instance, M. Abolins, R. L. Lander, W. Mehlhop, N. Huu Xuong, and P. Yager, Phys. Rev. Letters 11, 381 (1963).
[9] P. Freund (to be published).

F for $T=\frac{3}{2}$, $Y=-1$. As before, we indicate the heavier member of a pair by a prime.

$1\oplus 8$

No mass formula characterizes these five isotopic multiplets, but the inequality

$$\Lambda'\geq \tfrac{1}{3}(2N+2\Xi-\Sigma)\geq \Lambda\,, \tag{8}$$

must be satisfied.

There is a discrepancy from the Gell-Mann–Okubo formula for the eight baryons of about 5 MeV. We cannot attribute this discrepancy to the existence of a ninth (unitary singlet) baryon mixing with the Λ, because the inequality puts the mass of the ninth baryon *below* the known mass of the Λ, in contradiction with experiment.[10]

$8\oplus 10$

Without mixing, there are three mass relations for these multiplets; with mixing there are only two:

$$2\Xi'+2N-3\Lambda-\Sigma'=\Sigma-2\Xi+\Omega\,, \tag{9a}$$

and

$$(\Xi-\tfrac{1}{3}\Delta-\tfrac{2}{3}\Omega)(\Xi'-\tfrac{1}{3}\Delta-\tfrac{2}{3}\Omega) = (\Sigma-\tfrac{2}{3}\Delta-\tfrac{1}{3}\Omega)(\Sigma'-\tfrac{2}{3}\Delta-\tfrac{1}{3}\Omega)\,. \tag{9b}$$

When mixing is absent, both the left-hand sides and the right-hand sides of these equations individually vanish. The inequalities that ensure that mixing levels repel are

$$\Sigma'\geq \tfrac{1}{3}\Omega+\tfrac{2}{3}\Delta\geq\Sigma\,, \tag{10a}$$

and

$$\Xi'\geq \tfrac{2}{3}\Omega+\tfrac{1}{3}\Delta\geq\Xi\,. \tag{10b}$$

These are not independent.

It is simple to show that Δ, Σ, and Ξ satisfy an equal-spacing rule if and only if there is no mixing.

$10\oplus 27$

This case may have relevance to the understanding of the $j=\frac{3}{2}^+$ meson-baryon resonances, since it is believed[11] that the forces which give rise to the resonances in the decuplet may also be attractive in the 27-plet.

Without mixing, the decuplet masses may be described by two parameters and the 27-plet masses by three; there are two mass relations for the former (equal spacing) and six for the latter. Mixing, which affects only the Σ, Ξ, and Δ states common to both multiplets, introduces a sixth parameter which reduces the total number of mass formulas to seven. The surviving mass relations are:

(a) Three relations among the unmixed components of the 27-plet, unchanged by the mixing.

(b)
$$-(\Xi+\Delta-2\Sigma)=(\Xi'+\Delta'-2\Sigma')\,, \tag{11a}$$

$$\Omega+\Sigma-2\Xi=2\Xi'+2N-\Sigma'-3\Lambda\,. \tag{11b}$$

(c)
$$[\Delta-\tfrac{1}{4}(3F+7N-6\Lambda)][\Delta'-\tfrac{1}{4}(3F+7N-6\Lambda)] =5[\Xi-\tfrac{1}{4}(F-3N+6\Lambda)][\Xi'-\tfrac{1}{4}(F-3N+6\Lambda)] =15/8[\Sigma-\tfrac{1}{2}(N+F)][\Sigma'-\tfrac{1}{2}(N+F)]\leq 0\,. \tag{11c}$$

Again, each side of these relations vanishes in the absence of mixing. In this case, however, the Δ, Σ, and Ξ masses can remain equally spaced even in the presence of much mixing. Such equal spacing implies that the Σ, Ξ, and Ω masses are also equally spaced.[12] Thus, if the $J=\frac{3}{2}^+$ baryon decuplet does in fact mix with a heavier $J=\frac{3}{2}^+$ 27-plet, we can still deduce the mass of Ω from the equal spacing of Δ, Σ, and Ξ in this scheme.

Identification of Σ(1660) as a possible element of the 27-plet which mixes with Σ(1385) is not compatible with these relations unless an additional low-lying Ξ′ or Δ′ exists, since equal spacing for Δ, Σ, Ξ implies a corresponding equal-spacing law for Δ′, Σ′, and Ξ′.

ACKNOWLEDGMENT

Our interest in this problem was much stimulated by conversations with Professor J. Schwinger.

[10] In a recent theory of J. Schwinger [Phys. Rev. Letters **12**, 237 (1963)], a ninth baryon is predicted. The small departure from the Gell-Mann–Okubo formula for the low-lying eight baryons is there attributed to higher order effects, not to mixing.

[11] R. Cutkosky, Ann. Phys. (N.Y.) **23**, 415 (1963).

[12] As indeed they are; V. E. Barnes, P. L. Connolly, D. J. Crennell, B. B. Culwick, W. C. Delaney *et al.*, Phys. Rev. Letters **12**, 204 (1964). This is a general result: If we have four levels that would be equally spaced in the absence of mixing, and if three of them are equally spaced, then so is the fourth, even if there is mixing. We can show this for the mixing of any two representations, but since we have shown it explicitly for the only two cases of physical interest, we will forbear from giving the proof.